**Linear Algebra**

**1.1 LINEAR SYSTEMS**

**Linear** **Equation** is the equation of a straight line on a graph.

**Linear** **System** is a collection of linear equations with a finite set of variables. They can have **1**, **0** or an **infinite** number of solutions.

**Systems of linear equations** can be represented by a **matrix**. An **augmented** **matrix** is the complete system, and the **coefficient** **matrix** is solely based on the statements not their equated values.

**Elementary Row operations** are operations which are used to solve a matrix and change it into **RE** and **RRE**. These include:

**Multiplying a row by a nonzero scalar value (n\*Ri)**

**Interchanging two rows (Rj🡨🡪Ri)**

**Adding a multiple of one row to another (Rj + nRi)**

**1.2 ELIMINATION**

There are two important forms for a matrix. **Reduced Row Echelon (RRE) and Row Echelon (RE) form.**

**Row Echelon –** All the leading ones of the matrix have been formed and all values beneath a leading one are 0’s.

**Reduced Row Echelon –** All the leading ones have 0’s beneath and below them. Any rows of 0’s are at the very bottom of the matrix.

Values that are left in the equation that aren’t leading after the matrix has been brought to RRE are called **free** **variables**. The columns that have leading ones are called **leading** **variables**. The leading ones refer to the variable in that position and are equal to the row.

|X Y W Z| n |

| **1 0 0 0** | 4 |

| **0 1 2** **0** | 3 | Bolded value is the **coefficient** **matrix**.

| **0 0 0 1** | 2 |

As can be seen this matrix is in RRE and has **three leading variables** and **1 free variable.**

**X = 4 Y = 3 – 2W Z = 2**

X, Y, Z are Leading Variables because they have leading 1’s in their spots, while W is a free variable because it does not have a leading 1.

When you have one or more free variables (Called **parameters**) it is a sign that there are an **infinite** **number** **of** **solutions** because those parameters can be set to any value and the system will be solved. So the solution set has infinite possibilities.

If there are parameters in the reduced row echelon form and there is the possibility of solution then you have infinite numbers, however if you have no solution you might still see parameters. The only time that there are **no solutions** is if in the **augmented matrix** there are more leading **1’s** than in the **Coefficient matrix**.

| **1 0 0 :** 1 |

| **0 1 0 :** 2 | As can be seen here: There are more leading ones in the

| **0 0 0 :** 1 | augmented matrix than in the coefficient matrix (Bold)

There are special systems of equations called **Homogeneous** **Systems**. These are unique in the fact that they have a **trivial** **solution**. This is that all the **equations equal 0**. This means that these systems will always be **consistent** (have **one or more solutions**), and as such cannot have no solution.

| **1 0 0 0** **:** 0 |

| **0 1 0 0** **:** 0 | The rightmost column has all zeros which means it will

| **0 0 1 0** **:** 0 | at least have one solution.

| **0 0 0 1** **:** 0 |

**Theorem 1.2.1 - A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.**

**1.3 ARITHMETIC**

Matrices are essentially a **rectangular array of numbers**. These numbers represent external variables or values, and the individual numbers are referred to as **entries**.

It was desirable to find a way in which these matrices could be manipulated, and as such matrix arithmetic was developed to do just that.

**Addition**:

In order to add matrices they have to be the **same size (Rows and Columns).** So long as this requirement is met you just add **each corresponding entry together**.

**Subtraction:**

Must be the **same size**, and you just **subtract each corresponding entry**.

**Multiplication:**

There are **two** **types** of matrix multiplication. Firstly, **scalar** **multiplication** in which a scalar value is multiplied to the matrix and thus each entry is multiplied by the scalar. The second type is **matrix** **multiplication**. This is the multiplication of a matrix by another matrix. Although matrix multiplication has multiplication involved it is **not** **commutative**. Matrix multiplication is actually changing the values of the variables to be represented by other linear equations. This is done by multiplying the matrices as if they were a multiplication table.

**Multiplication Example**:

| 1 2 3 4 |

(B) | 2 3 4 5 |

| 3 4 5 6 |

| 4 5 6 7 |

| 1 2 3 4 | | (4\*4) + (3\*3) + (2\*2) + (1\*1) ….. …… ….. |

(A) | 2 3 4 5 | | (5\*4) + (3\*4) + (3\*2) + (2\*1) ….. …… ….. |

| 3 4 5 6 | | (6\*4) + (5\*3) + (4\*2) + (3\*1) ….. …… ….. |

| 4 5 6 7 | | (7\*4) + (6\*3) + (5\*2) + (4\*1) ….. ….. …… |

(AB)

**Multiplication Explanation:**

X Y

| 2 3 **:** 4 | 2X + 3Y = 4

| 3 1 **:** 13 | (A) 3X + Y = 13

**However if you set the variables in the coefficient matrix to equations:**

W Z

| 1 1 | X = W + Z

| 4 1 | (B) Y = 4W + Z

**This would be like saying (matrix multiplication – Replacing each instance of X and Y):**

W Z

| 1 1 |

|4 1 | (AB)

X Y W Z

| 2 3 | | (3\*4) + (2\*1) (3\*1) + (2\*1) | 🡪 |**14 5**|

| 3 1 | | (1\*4) + 3\*1) (1\*1) + (3\*1) | 🡪 | **7 4**|

**Or in the form of a linear expression:**

2(W + Z) + 3(4W + Z) = 4

3(W + Z) + 1(4W + Z) = 13

**Can be rewritten like this:**

(2\*1 + 3\*4)W + (2\*1 + 3\* 1)Z = 4

(3\*1 + 1\*4)W + (3\*1 + 1\*1)Z = 13

**Which when simplified is:**

14W + 5Z = 4

7W + 4Z = 13

**This is represented in matrix form as:**

| **14 5 :** 4 |

| **7 4** **:** 13 |

**1.4 INVERSES**

When multiplying Matrices you cannot say that AB = BA. **Most of the time AB != BA.** There are some cases where this is true however. If A is a square matrix and B is the same size as A, then **AB** = **BA** **so long** **as B = A-1****or** **B = I or B = 0**. If B = A-1 then **AB** = **AA**-1 = **I:** This is true because any matrix or number multiplied by its inverse equals 1. In matrices, the **Identity** **Matrix** (I) is the equivalent to 1 for numbers. The identity matrix is a square matrix in which all diagonal entries are 1 and the rest of the entries are 0. Anything multiplied by I is equal to itself. And finally, if B = 0 then it is a matrix that is filled with all entries equal to 0. This means that anything multiplied by a **0 matrix** act just like numbers do when multiplied by 0: they always equal 0.

AI = IA = A

A-1A = I

AB = I 🡪 A-1AB = A-1I 🡪 IB = A-1 🡪 B = A-1

AB = 0 🡪 A-1AB = A-10 🡪 B = 0

When multiplying Matrices it is important to note that if you **Pre**-**multiply** or **Post**-**multiply** a matrix to one side of an equation then you must do the same to the other side.

**Pre-multiply: AB = I 🡪 A-1AB = A-1I**

Note that when A-1 was multiplied to the left side in front of the equation it was done so on the right side in front of the equation as well.

**Post-multiply: AB = I 🡪 ABB-1 = IB-1**

Note that when B-1 was multiplied to the left side behind the equation it was done so on the right side behind the equation also.

**Theorem 1.4.1 – The following arithmetic operations are valid for matrices so long as the size is correct for the involved matrices:**

**1) A + B = B + A 2) A + (B + C) = (A + B) + C**

**3) A(BC) = (AB)C 4) A(B + C) = AB + AC**

**5) (B + C)A = BA + CA 6) A(B – C) = AB – AC**

**7) (B – C)A = BA – CA 8) a(B + C) = aB + aC**

**9) a(B – C) = aB – aC 10) (a + b)C = aC + bC**

**11) (a – b)C = aC – bC 12) a(bC) = (ab)C**

**13) a(BC) = (aB)C = B(aC)**

**Theorem 1.4.2 – The following is true for arithmetic involving 0 matrices:**

**1) A + 0 = 0 + A = A**

**2) A – A = 0**

**3) 0 – A = -A**

**4) 0A = A0 = 0**

Matrices can only have one unique inverse if it is invertible.

**If A is invertible and B and C are inverses of A then:**

**AB = I = BA 🡪 (CA)B = C(BA) 🡪 IB = CI 🡪 B = C**

When inverting matrices it can be shown that (AB)-1 = B-1A-1

**(AB)-1(AB) = I 🡪 (B-1A-1)(AB) = B-1(A-1A)B = B-1IB = B-1B = I**

**Theorem 1.4.6 – If A and B are the same size and are both invertible then:**

**(AB)-1 = B-1A-1**

When dealing with **exponents** and Matrices it can be said that the **rules** **of** **exponents** are the same as those for regular numbers:

**ArAs = Ar+s**

**(Ar)s = Ars**

**An = (A \* A \* A \* A …..), n number of factors**

**A-n = (A-1) = (A-1 \* A-1 \* A-1 \* A-1 \* A-1……), n number of factors**

When a matrix which is multiplied by a **scalar** **value** (**k**) is set to a **power** **of n**:

**(kA)n = kn(A \* A \* A….)**

**\*\*This is also true for inverses.\*\***

When dealing with **Polynomial** **Functions** a matrix can be placed instead of a variable(x, a, b, c,…)

**F(x) = x2 + x + 3**

can be shown with matrices as:

**F(A) = A2 + A + 3(I)**

**= AA + A + 3(I)**

You can **transpose** **a matrix** by switching the columns with the rows. This is shown with a T next to the Matrix.

| 1 0 2 | T | 1 3 6 |

| 3 1 4 | 🡪 | 0 1 1 |

| 6 1 8 | | 2 4 8 |

**Theorem 1.4.9 – The following things can be said about transposing matrices:**

**1) ((A)T)T = A**

**2) (A + B)T = AT + BT and (A - B)T = AT - BT**

**3) (kA)T = k(At)**

**4) (AB)t = BtAt**

**1.5 ELEMENTARY MATRICES**

**Elementary** **matrices** are matrices which can be found by applying a **single** **ERO** (elementary Row Operation) on an **Identity** **matrix**.

| 1 0 0 | | 1 0 0 |

| 0 1 0 | R2 \* 2 🡪 | 0 2 0 | **This is an elementary Matrix**

| 0 0 1 | | 0 0 1 |

**Elementary** **Matrices** **are** **special** because the product of a Matrix and an Elementary Matrix is the same as the ERO that was performed on the Elementary Matrix being applied to the Matrix.

A = |1 2 3| E = | 2 0 0 | EA = | 2 4 6 |

|1 0 0| | 0 1 0 | | 1 0 0 |

| 0 0 1 |

Another interesting thing about Elementary Matrices is that they can be used to **find an inverse**.

If you **place the A matrix next to the I matrix** you can derive the Inverse by **applying** **ERO** to the Augmented matrix.

[A] = | 1 0 | [I] = | 1 0 | [A:I ]= | 1 0 1 0 | 🡪 [I:A-1] = | 1 0 : 1 0 |

| 2 3 | | 0 1 | | 2 3 0 1 | ERO | 0 1 : -2 1/3|

A-1 = | 1 0 |

|-2 1/3|

In order for a matrix to be **invertible it must be a square matrix**. An invertible matrix when in **RRE form is an I matrix.**

**1.7 DIAGONAL, UPPER AND LOWER TRIANGULAR, AND SYMMETRIC MATRICES**

**Diagonal Matrices** are matrices in which all entries except the main diagonal are 0.

**Upper Triangular Matrices** are matrices in which all entries below the main diagonal are 0.

**Lower Triangular Matrices** are matrices in which all entries above the main diagonal are 0.

**Symmetric Matrices** are matrices in which all corresponding entries about the main diagonal are the same as their mirrored positions.

**2.2 DETERMINANTS**

The **Determinant** is a **unique** **number** that is **associated** **with** **a square matrix**. It can be calculated by reducing a matrix to RE form, or from an UPPER or LOWER TRIANGULAR Matrix.

**If the Matrix is UPPER or LOWER TRIANGULAR then:**

Det(A) is the product of the numbers in the **main** **diagonal**.

**If the matrix needs to be reduced to RE form then:**

Det(A) is the product of numbers in the **main** **diagonal** in **RE** **form** and numbers based on the **ERO** you applied to the matrix to get it to RE:

**1 – If you divide a row by a scalar (k) then multiply the Det(A) by k**

**2 – If you switch rows then multiply Det(A) by -1**

**3 – If you add a multiple of a row to another then multiply Det(A) by 1**

If the **Determinant of a matrix is 0** then the matrix **cannot be inverted**; a **determinant of 0** means that either the **matrix has no solution** or **an infinite number of solutions**.

**Example:**

| **1** 4 5 |

Det(A) = Det | 0 **2** 6 | = (**1 \* 2 \* 3**) = **6**

| 0 0 **3** |

As can be seen, this matrix is in Upper Triangular form and thus we can find the determinant by multiplying the entries in the main diagonal.

| **2 2 2** | | 1 1 1 | | 1 1 1 | |**1** 1 1 |

Det(A) = Det | 0 3 6 | = (**2**)| **0 3 6** | = (**2\*3**) | 0 **1 2** | = (**2\*3\*1**) | 0 **1** 2 | = (**2\*3\*1**)\***2** = **12**

| 0 3 8 | | 0 3 8 | | 0 **3 8** | | 0 0 **2** |

As was seen in the example above, when a factor was taken from a row, that value was multiplied to the determinant, and when a multiple of a row was added or subtracted then it was multiplied by one, and had a row been switched with another it would have resulted in the determinant being multiplied by -1. These ERO’s were used until the matrix had turned into RE form. From this point the determinant was then multiplied by the main diagonal entries.

**2.3 – PROPERTIES OF THE DETERMINANT FUNCTION**

As was mentioned earlier, the determinant is used to determine if a matrix is invertible or not.

In order to understand determinants we have to understand who they are related to the square matrix and how they related to other determinants.

**Det(kA) = kn \*det(A)**

This is like so because when a matrix is multiplied by a scalar each row gets a factor of k applied to it, which when finding the determinant is taken out of each row. Thus if you have 3 rows, you would take out k three times, and then it would be k\*k\*k or k3 which is then multiplied by the remaining determinant.

An important thing to note is that there is not a relationship between det(A), det(B) and det(A+B).

**Det(A) + Det(B) ≠ Det(A+B)**

However, if only one row or column in two nxn matrices is the different between the two, then the sum of their determinants is equivalent to a matrix with the same rows or columns, and the sum of the other row or column:

**Det(A) + Det(B) = Det(C) When:**

**A** **| a b | + B| a b | = C| a b |**

**| c d | | e f | | c+e d+f |**

An important theorem in the foundation of determinants is that:

**Det(AB) = Det(A)Det(B)**

This is shown by the fact that **in order for a matrix to be invertible** **Det(A) ≠ 0**, which means that if **Det(A) = 0 then it is not invertible** which also means that AB is not invertible. Because if Det(A) = 0, Det(AB) must also equal 0. So if **A is invertible then that means that it is made up of Elementary matrices**, and since the determinant of an EM is equal to the ERO, so **Det(A) = Det(E1E2E3…)**, and then that means that **Det(AB) = Det(A)Det(B)**.

Another thing is that:

**Det(A-1) = 1 / Det(A)**

This is because **Det(A-1A) = Det(I) = Det(A-1)Det(A) = 1 -** This means that Det(A-1) = 1/ Det(A)

**2.4 – COFACTOR EXPANSIONS AND CRAMER’S RULE**

**COFACTOR EXPANSION:**

**Cofactor expansion** is a method for determining the determinants of square nxn matrices.

Given an nxn matrix it is possible to figure out the determinant by multiplying the entry(j) of row(i) with determinant(**minor**) based on removing row(i) and column(j) and calculating the it remaining entries.

**| a b c | = a \* | e f | + (-1)b \* | d f | + c \* | d e |**

**Det(A) = | d e f | | h I | | g I | | g h |**

**| g h I |**

This is known as **Cofactor Expansion**, and the determinants of in the cofactor multiplication are known as **minors**. The **Minor of A11** multiplied by a in the above calculation. Notice that in the second term of the equation, there is a **(-1) being multiplied with the term**. This has to do with the theory that **cofactors (Minors) can only be calculated on the entry A11** so you have to **account for the ERO for row switching which multiplies the determinant by (-1).** Thus all entries have to be accounted for based on the # of row switches needed.

**COFACTOR MATRIX:**

Since you can calculate the cofactors for each entry in a square matrix, you can find the cofactor for every entry and then put them in a **Cofactor Matrix.** Each entry **(Aij)** in a square matrix has a corresponding cofactor **(Cij)**. Thus the matrix looks like:

**| Cij . . Cin|**

**Cof(A) = | . . . . |**

**| Cnj . . . |**

**ADJOINT MATRIX:**

The **adjoint of A is the transpose of the cofactor matrix.**  With this knowledge we can define the following equation:

**A-1 = Det(A-1)Adj(A) = 1/Det(A) \* Adj(A)**

**CRAMERS RULE:**

**Cramer’s Rule** is an algorithm that states that a solution to Ax = b can be found by replacing the **ith column in A** by **b**. Then calculating that new Matrices determinant Det(Ai) and dividing it by Det(A). The easiest way to do this is to use Cofactor Expansion for the matrices.

**Xi = Det(Ai)/Det(A)**

**3.1 – Introductions to Vectors**

**Vectors** can be represented in two ways**: Geometrically and Algebraically.**

**GEOMETRIC VECTORS:**

Geometrically vectors are represented as directed line segments or arrows. The direction is represented by the orientation of the arrow, and the magnitude of the vector is represented by the length of the arrow. The tail of the vector is known as the **Initial Point**  and the head of the vector is known as the **Terminal Point.** Vectors are denoted by lowercase letters boldfaced.

Ex. **v, w, k, u, x, etc…**

**Scalars** are numbers that typically are multiplied to vectors. They are usually denoted as italic lowercase letters.

Ex. *v, w, k, n, d, etc…*

Vectors are **equivalent** when their **direction and magnitude** are the same, this means that equivalent vectors can have different initial and terminal points. We then can say that two vectors are equal even if they are at different positions.

Vectors are said to be added together when the tail of one vector (**u**) is added to the head of another vector (**v**) and then, the corresponding vector that is made up between the head of (**u** and tail (**v**). It can said then that:

**u + v = v + u**

This is known as the **law of parallelograms.** Because when **u + v** and **v + u** are placed together they form a parallelogram.

When a Vector of length 0 is added to another vector, it has no effect:

**0 + u = u + 0 = u**

When subtracting vectors (**u – v)**, it can said that it is the same as adding the **negative of v.** Since all vectors have a direction the negative of a vector is merely a vector with the same magnitude but in the opposing direction.

**u – v = u + (-v)**

This can be geometrically represented by placing the heads of both vectors together and forming a vector between their tails.

You can also multiply a vector by a scalar. This is known as **Scalar Multiplication.** This is said to be a new vector with the direction of the original (unless multiplied by a negative), but a magnitude |*k*|times larger.

**|***k***|u = w**

or if **negative scalar** then:

**|***k***|(-u) = w**

**ALGEBRAIC VECTORS:**

We have seen how vectors are shown geometrically, but we can also represent them algebraically. This is done by representing a vector by its coordinates otherwise known as components.

**2-Space Vector: v = (**v1 , v2**)**

**3-Space Vector: w = (**w1 , w2 , w3**)**

Where the components v1 , v2 , etc…are the coordinates relating to the axes.

If two vectors are equivalent that means that their respective components must be equal. Thus if you had two vectors **v(**v1 , v2**)** and **w(**w1 , w2**),** then in order for them to be equal:

v1 = w1 and v2 = w2

As expected, when adding two vectors it can be said:

**v + w = (**v1 + w1 , v2 + w2**)**

and multiplying a vector by a scalar is:

*k***v = (***k*v1 , *k*v2**)**

Subtraction can be explained in a similar fashion:

**v - w = (**v1 - w1 , v2 - w2**)**

**VECTORS IN 3-SPACE:**

When talking about vectors in 3-space we refer to it in regards to three coordinates (components); these are the x, y, and z positions. Any vector in 3-space is composed of a component in each of these directions. This form of coordinates is known as a **rectangular Coordinate System.** It is made up of three planes: **xy-plane, xz-plane, and yz-plane.** There are two ways to express a rectangular coordinate system: **right-handed** and **left-handed.**

**Right-Handed:** If you were to place a screw in the direction of the z-axis it would advance in that direction.

**Left-Handed:** if you were to place a screw in the direction of the z-axis and turn it, it would retract.

Most of the time a vector will not be placed at the origin at which point in order to find the vector, you have to take the **initial point and subtract it from the terminal point**. This will be equal to the vector:

Points P1(2,3) and P2(4, -5), then the Vector would be equal to:

= P1 – P1 = (4 – 2, -5 – 3) = (2, -8)

**3.2 – NORM OF A VECTOR AND VECTOR ARITHMETIC:**

**Theorem 3.2.1 – Properties of Vector Arithmetic**

**a) – u + v = v + u b) - (u + v) + w = u + (v + w)**

**c) – u + 0 = 0 + u = 0 d) - u + (-u) = 0**

**e) –** *k***(***l***u) = (***kl***)u f) -** *k***(u + v) =** *k***u +** *k***v**

**g) – (***k* **+** *l***)u =** *k***u +** *k***v h) -** 1**u = u**

**NORMS OF VECTORS:**

The **length** of a vector is called the **norm of a vector.** It is denoted by **|| u ||** it is calculated using **Pythagoras Theorem.** You just **square root the sum of the components’ squares.**

**|| u ||** =

And the same goes for 3-space Vectors.

The norm can also be said to be the distance between two points as it is the length of the vector formed between them.

**3.3 – DOT PRODUCT AND PROJECTIONS**

**DOT PRODUCT:**

The **dot product** or otherwise known as **the Euclidean Inner Product** is denoted as **u∙v,** and is defined as:

**u∙v = ||u|| ||v|| cosθ**

This is due to its relationship with the Cosine Function. Given two Vectors (**u(u1 , u2)** and

**v(v1 , v2) )** and their respective angles( **θu , θv** )in relation to the x-axis (where the angle between them is **Θ**): it can be said that:

**Cos(Θ) = Cos(θu – θv) = Cos(θu)Cos(θv) + Sin(θu)Sin(θv)**

**= ( u1 / ||u|| )\*( v1 / ||v|| ) + (u2 /||u|| )\*(v2 / ||v|| )**

**= ( u1v1 ) / (||u|| \* ||v|| ) + (u2v2) / ( ||u|| \*||v|| )**

**= ( u1v1 + u2v2) / ( ||u|| \*||v|| )**

**= ( u∙v ) / (||u|| \* ||v|| )**

Using this knowledge it is now possible to find the angle between two vectors, as well as their length or their dot product, so long as you have 2 pieces of information.

**PROJECTIONS**:

One application of the dot product is in the use of **Projections.**

**Projection –** Is the **component of vector u parallel to the vector a.** A projection is denoted as **projau** and is calculated as:

This is like this because when it is simplified it becomes:

In other words it is equal to the **ratio of the length of the parallel component and the length of direction unit**. You can also find the **orthogonal component of the vector**:

**DISTANCE:**

Projections can be used to find the **shortest distance between a point P(xp , yp) and a line**

**L(ax + by + c = 0)** which has a normal **n(a , b)** where **Q is a point on the line.**

This is possible because since the point Q is on the line which means that **c = –axQ – bxQ** which is also the dot product of and **n.**

**CROSS PRODUCT:**

The cross product is defined as the vector product of two vectors that is **perpendicular to both of the vectors**. It is calculated by the following equation:

An important thing to remember is the orientation of the cross product is based on the **right hand rule**. That is the direction of your fingers when you hold your hand out, dictates the order in which you take the product, and your thumb points in the direction of the vector product.

**Ex. If your thumb is pointing up, and u comes before v then you go: u x v = w(in the up direction)**

**AREA:**

While the dot product was used to find the length of a projection, and the distance between a point and a line, the **cross product can be used to find the area of a parallelogram** made by two vectors. The area is equal to the **magnitude or norm of the cross product**:

**||u x v|| = ||u|| ||v|| sinθ**

**SCALAR TRIPLE PRODUCT:**

If you take the dot product of a cross product with a vector, you get what is called a **scalar triple product**. This is equal to the **volume of the parallelepiped made by the three vectors**.

**u ∙ (v x w) =**

**PLANES IN 3-SPACE:**

In 3-Space planes are represented by linear equations in the form of:

**ax + by + cz +d = 0**

There are a few ways to represent the equation of a plane: **point-normal** and **vector-form.**

**Point-Normal Form –** This form is expressed by taking the dot product of the normal of the plane **n**, to a vector lying on the plane between two points (**P0P1**).

**Vector Form –** This form is expressed using vectors instead of Points. It is expressed by the dot product of the normal **n,** and the vectors (**r** and **r0)** where **r** and **r0** are the vectors between the origin and two points that lie on the plane respectively.

These equations give the positions of all points that lie on the plane. And knowing this it is possible to figure out the **equation** **of a plane based on three or more points** (since there are three variables). To do this just place the coordinates of the three points in their respective places and **solve the linear system** via the use of a matrix. This will allow you to solve for the values of a, b, c, and d.

**LINES IN 3-SPACE:**

In 3-Space lines can be expressed in two forms: **Parametric Equations** and **Vector Equations.**

**Parametric Equations –** These are expressed separately for each of the coordinates x, y, and z. They are in the form of: **x = xp + a*t***

**Vector Equations –** This form is expressed based on the vectors from the origin to two Points and stating that their difference is a scalar multiple of a vector parallel to the line.

**DISTANCE BETWEEN PLANES:**

Similar to how the distance between a point and a line was calculated previously, the distance between a point and a plane can also be calculated. Given a normal to the plane **n(a,b,c),** an arbitrary point **Q(xq , yq , zq)** and a point **P(x, y, z)** the distance between the plane and the point **P** can be calculated as:

This is like so because it is related to the projection of the vector between a point on the plane **Q** and point **P** on the normal of the plane **n**. Since the point **Q** must satisfy the equation **ax + by + cz = -d** then d must equal:

Which allows us to substitute **d** into the equation and solve it.

**Chapter 10 - Complex Numbers:**

**IMAGINARY AND COMPLEX NUMBERS:**

Any real number when it is squared becomes positive. However, what happens when the square of a number is not positive? This is known as an imaginary number. There is a special character denoted as that is defined as:

These numbers are called imaginary because they do not exist in nature, however they do have applications involving systems that oscillate, so they do need to be understood.

When a number has a real and imaginary component they are called **complex numbers.** Some examples include:

**COMPLEX ARITHEMETIC:**

Most forms of arithmetic work with complex numbers as they do with real numbers however they can sometimes be trickier due to the presence of .

As can be seen when complex numbers are added the **real** **components** **are** **added** **together** and the **imaginary** **components** **are** **added**. This is very similar to how matrices are added together. This is also similar for **scalar** **multiplication**:

Another thing that can said about scalar multiplication is that a complex number is said to be the negative of z if:

This similarity to matrix algebra does not however exist for **complex** **multiplication**. Multiplication of complex numbers is similar to multiplying binomial equations:

As can be seen above in the examples it is exactly like foiling, except that instead of having variables you have numbers, and remembers that . Another thing to note is that the number that is the **product** **is** **also** **a complex number**. There are a few instances in which it will be a real number though.

**COMPLEX PLANE:**

Before we look at that though it would be best if we show a different way in which to express complex numbers. It can be said that a complex number can be expressed as where the coordinate can be plotted on the **complex plane.**

**Complex Plane –** The complex plane is set up in a way where the x-axis is the real axis(the real component) and the y-axis is the imaginary axis(the imaginary component).

**MODULUS AND ARGUMENT:**

Now since we can imagine a complex number as a point on the complex plane that means it could also be imagined as a vector, and as is known vectors have both a **length** and **direction.**

**Length –** The length of a complex vector is known as its **modulus,** and it is expressed as the hypotenuse of the triangle formed by the real and imaginary components and is denoted .

**Direction –** The direction of the complex vector is denoted by the angle between the real axis and the vector. The angle is referred to the **argument,** and the **Principle Argument** is the angle:

Note that the Principle Argument is the denoted with the **capital letter A** and it has the particular range. This is because as it is known there are infinite number of angles that could represent the orientation.

As would be expected the length is a real number, and as was mentioned earlier there are some instances when a real number is the product of two complex numbers. This is when a complex number is multiplied by its **conjugate.**

The conjugate of is denoted as , and as can be seen it has the relationship:

There is an important point to make about conjugates though. The conjugate of a complex number is said to have an **opposite signed imaginary component.** That means that only the imaginary component has a different sign and not the real component.

**COMPLEX DIVISION:**

Now it is time to focus on the process of **division of complex numbers**. Now, it can be said that division is the inverse of multiplication which means that if we multiply the quotient of a complex number by its product we should get the original complex number.

This makes it a lot easier to figure out the quotient of two complex numbers because now it is merely multiplication of complex numbers, and division of a scalar.

Now that arithmetic has been covered for complex numbers it is now possible to apply this to solving linear systems that contain complex numbers.

**POLAR COORDINATES:**

Up to this point we have glanced briefly at complex numbers in the complex plane, and this plane was set up to feel very much like the Cartesian coordinate system, but they can also be shown in the polar form, which for some applications can be easier to work with.

**Polar Form –** This is a coordinate system that focuses on the length of a vector, and the angle. So for complex numbers it is referred to by its **modulus** and its **Argument.**

When it is placed in an equation it is in the form of:

Where is a complex number, and **.**

There is a process by which you convert the Cartesian coordinates to polar form:

Example:

1 – Find the modulus:

2 – Write the equations out for **a** and **b:**

3 – Figure out which angle satisfies both equations:

4 – Rewrite it all out in polar form:

This will work for every instance and you can just reverse it to figure out the Cartesian form. Just figure out what **a** and **b** will be and then rewrite it.

**GEOMETRIC INTERPRETATION OF COMPLEX NUMBER ARITHMETIC:**

We know how to do multiplication in the Cartesian form, however we do not know how it relates to polar. We could just convert to Cartesian, do the multiplication, then convert back, but that wont always be nice, and sometimes it can lead to awkward angles, so it is best we can find a relationship in polar form.

The same works for division:

**ROOTS AND POWERS OF COMPLEX NUMBERS:**

It can be shown that if a complex number is multiplied by itself n times then:

Knowing this knowledge can help calculate the roots of a complex number.

Root – The nth – root is said to be a number such that:

It can then be said that:

The remaining roots can be found by adding to the angle n-1 times.

This is so because if we imagine a complex number as a vector than we can imagine that it is enclosed by a circle of radius and the root angles are those by which 360 degrees can be divided n times about the initial angle of vector z.

There is another way to represent complex numbers in polar form:

Where:

We can also express **conjugates** of z in **polar** **form**:

**Chapter 7 - Eigenvalues and Eigenvectors:**

**EIGENVALUES:**

As was mentioned earlier an eigenvalue is said to be any value which when multiplied by a vector is equal to a matrix being multiplied by the same vector:

When this is true, is said to be an eigenvalue of .

**FINDING EIGENVALUES:**

Finding eigenvalues is important and as such there is a way to do so. It can be said that:

This is known as the **characteristic equation of** This means that if we find the determinant we can solve the equation for .

This expression is called the **characteristic polynomial of**

The degree of this polynomial will be equal to the number of columns and rows of A.

**INVERTIBILITY AND EIGENVALUES:**

Since the determinant when factored will be in the form:

It can be said that in order for A to be invertible since that would mean that the determinant of A is 0.

**TRIANGULAR MATRICES:**

When a matrix is **Triangular** it can be said the determinant is the product of the diagonal entries. This is also true for **eigenvalues**:

Where is an entry on the diagonal of a triangular matrix.

So it can be said that for **triangular** matrices the **eigenvalues** **are** **the** **diagonal** **entries**.

**EIGENVECTORS:**

It can be said that if is an eigenvalue of A, then **x** is an eigenvector such that**.**

It is then quite evident that **for every eigenvalue**, there is an **associated eigenvector x.**

For a vector to be considered an **eigenvector** it must be a **non-zero vector** that satisfies the above equality.

**FINDING EIGENVECTORS:**

Now that we know what an eigenvector and eigenvalues are, we should learn how to find them. We know that in order to find the eigenvalues we have to solve the characteristic polynomial for. And we know that but how can we use this to figure out eigenvectors? Well we say that:

So this means that we know that is a **solution** **space** for the **eigenvector**. This is known as the **solution space** for **x** and since we are dealing with eigenvectors it is called the **eigenspace** for **x.**

And it found by solving the equation which in turn will give us the eigenvector for that particular eigenvalue. **Note that for every eigenvalue there is a corresponding eigenvector: this means you have to solve the system for each eigenvalue.**

Example: Find the eigenvectors corresponding to the eigenvalues: for the matrix:

So for the first eigenvalue the matrix is:

In order to get the eigenvector for we have to **solve** **the** **system**:

Which when solved yields:

It can then be said that:

The **eigenvector** for can then be said to be:

Where the **vector** **space** for is:

When the system is solved for the **vector** **space** is:

So the **eigenvectors** are said to be:

and

Note that has **two** **eigenvectors**. An eigenvalue can have up to as many eigenvectors as there are instances of itself in the characteristic polynomial. Also note that these two are **linearly** **independent** (not scalar multiples of one another) which means that they form the **basis** (a combination of linearly independent vectors) of the **vector** **space** for.

One thing that needs to be remembered is that:

**DIAGONALIZING MATRICES:**

When dealing with matrices it is said to be important to know whether there exists a Matrix P such that for an Matrix A:

Where D is a **diagonal** **Matrix** where each entry is an **eigenvalue** **for A**.

It can then be said:

Which if:

Then:

Which means that for each column in both AP and PD we have which is equivalent to **.**

This must mean that columns  **are in fact eigenvectors corresponding to .** Which means that if we want to find P which diagonalizes A all we have to do is **find** **the** **eigenvectors** **of** **A** and place them into a matrix. However, this matrix must be invertible, and as such must be square and each **column**/**row** must be **linearly** **independent**.

In order for an to be **diagonalizable** it must have**linearly independent eigenvectors**.

**FINDING MATRIX WHICH DIAGONALIZES A:**

In order to **find Matrix P:**

1 – Find all **eigenvalues** **for A**

2 – Find corresponding **Eigenvectors** **of A**

3 – Place **eigenvectors** into **columns** **of** **P**

4 – Check and make sure that

**MULTIPLICITY:**

There are terms that go with the idea of repeated eigenvalues, and the number of eigenvectors in an eigenspace. These are **Geometric Multiplicity** and **Algebraic Multiplicity.**

**Geometric Multiplicity –** This is the **dimension (# of eigenvectors) of an eigenspace associated with**  , so if there exists 2 eigenvectors associated with then the Geometric Multiplicity of is 2.

**Algebraic Multiplicity –** This is the **# of times that appears as an eigenvalue of A**, in other words the number of times that appears as a factor in the characteristic polynomial of A.

**POWERS OF MATRICES:**

One reason that we are interested in the diagonal of a matrix is because we can then use it to calculate the power of a matrix. The reason that this is like this is because:

Which means that:

So if we know D we can calculate :

This is very helpful since it is much easier to calculate than since it is diagonal which means we can just calculate each entry’s power **(** then multiplying it by **P**.

**Chapter 4 – Linear Transformations**

**LINEAR TRANSFORMATIONS AND LINEAR OPERATORS:**

It is important to be able to express linear transformations in different forms in order to make it easier to work with them. An important type of transformation is expressing a **point** or **vector** which is in **n-space** in terms of **m-space.** These are known as transformations fromand are denoted as **.**

**Linear Transformations –** These are transformations which change a vector (or point) to another vector (or point)

**Linear Operators –** These are transformations which apply changes to a vector to another vector.

As can be seen **Transformations express a vector in different dimensions**, while **operators change a vector in its current dimensions**. It can be said that if **x** is the vector being transformed, then **w** is the vector after the transformation, so in terms of vectors and matrices it can be shown:

Where A is called the **standard Matrix** for the transformation **T** which is **Multiplication by A.** This can also be expressed in different notation such as:

or

**GEOMETRIC EFFECTS OF LINEAR TRANSFORMATIONS:**

It can be said that whether a the column matrix x represents a point of vector, the transformation will be the same. Some important transformations are:

**Zero Transformation –** This is a transformation which **maps** **each** **point**/**vector** **to** **the** **0 point**/**vector**.

**Identity Operator –** This is a transformation which **maps** **each** **point**/**vector** **to** **itself**. This is technically an operator as it maps each vector into the same n-space.

**Projection Operator –** This is an operator that maps each **vector**/**point** **onto** **one** **of** **the** **axes**, **or** **planes**.

Example (projection onto x-axis):

As can be seen the x value is kept, and the y value is reduced to 0. So it is keeping only the x component which is the projection onto the x axis.

**Reflection Operator –** This is an operator that **reflects** **each** **vector**/**point** **about** **an** **axis**.

Example (Reflection about y-axis):

So it is shown that a reflection operator negates whatever the component it is that it is reflecting while keeping the other operators the same.

**Rotation Operator –** This is an operator that **rotates** **each** **vector**/**point** **about** **the** **origin**.

Example (Rotation 90 degrees CCW):

As can be seen that when you rotate you actually use cos() and sin() to map the vectors. These rotations are **CCW if the angle is positive** and **CW if the angle is negative**.

**Dilation/Contraction Operator –** This is an operator that **dilates** **or** **contracts** a vector either closer or farther away from the initial point.

While we only looked at operations on there are operators that apply transformations to as well. **Just remember to think about what is happening geometrically.**

On a note regarding **rotations in 3-space**, the rotations are imagined to be rotating around an axis known as the **axis of rotation.** The general standard matrix for rotation in 3-space around the rotation matrix **u** (a,b,c) is:

**COMPOSITIONS OF LINEAR TRANSFORMATIONS:**

It can be said that multiple linear transformations can be done in one standard matrix, and the reason that this is is because:

As can be seen the standard matrix for the is merely the multiplication of matrix B with A.

**Note that order is important. The order of transformations is seen as inside-outside.**

**Chapter 9 – More Linear Transformations**

**LINEAR TRANSFORMATIONS ON R2:**

There are more transformations that can be applied:

**Expansion/Compressions –** These are transformations that reduce one of the components of a vector.

As it can be seen it is scalar multiplication on one component.

**Shears –** These are transformations that slide a component as a function of the other component.

As it can be seen the translation is a function of the other component and the larger the component the larger the translation.

It can be said that if the **standard** **matrix** **is** **invertible** then all transformations by this matrix in 2-space are transformations of **elementary** **matrices** (**shears**, **expansions**/**compressions**, and **reflections**)

**Note that rotations and projections are not elementary matrices.**

**FINDING THE IMAGE OF A LINE:**

It is important to be able to find the **equation** **of the** **image** **of** **a line** after multiple transformations. The way in which to do this is:

Given find the image of the line using the Matrix A:

Which when solved for and:

Then substitute them into the original equation of the line:

This is the equation of **image** **of** **the** **line**.